Chapter 3

Vectors

Vectors

- Quantities which indicate both magnitude and direction
- Examples: displacement, velocity, acceleration, force, momentum, torque, angular momentum, electric field

Scalars

- Quantities which indicate only magnitude
- Examples: time, distance, speed, temperature

Arrows are used to represent vectors.
- ➤ The length of the arrow signifies magnitude
- ➤ The head of the arrow signifies direction
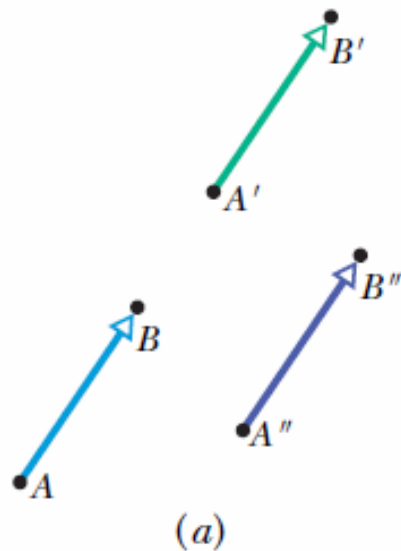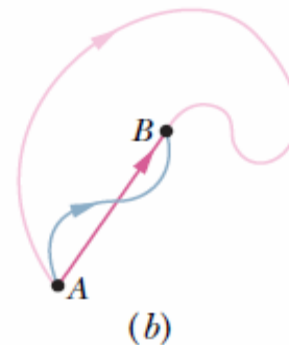- ➤ http://www.physicsclassroom.com/mmedia/vectors/vd.cfm

Fig. 3-1 (*a*) All three arrows have the same magnitude and direction and thus represent the same displacement. (*b*) All three paths connecting the two points correspond to the same displacement vector.

Sometimes the vectors are represented by bold lettering, such as vector **a**. Sometimes they are represented with arrows on the top, such as $\vec{a}$

3.3 Adding vectors geometrically

- Vector a and vector b can be added geometrically to yield the resultant vector sum, c.

$$\vec{s} = \vec{a} + \vec{b},$$

- Place the second vector, **b**, with its tail touching the head of the first vector, **a**. The vector sum, **s**, is the vector joining the tail of a to the head of b.

- Vector addition:

To add $\vec{a}$ and $\vec{b}$, draw them head to tail.
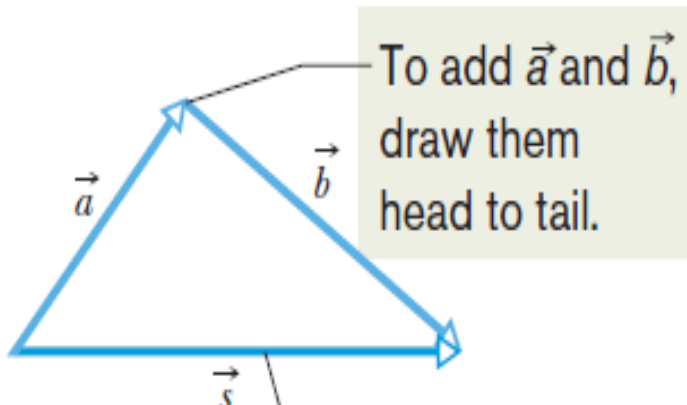
3.3 Adding vectors geometrically

- Some rules:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{(commutative law)}.$$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad \text{(associative law)}.$$

$$\vec{b} + (-\vec{b}) = 0.$$

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b}) \qquad \text{(vector subtraction)}$$

- Example: http://www.physicsclassroom.com/mmedia/vectors/ao.cfm

3.3 Adding vectors…sample problem

In an orienteering class, you have the goal of moving as far (straight-line distance) from base camp as possible by making three straight-line moves. You may use the following displacements in any order: (a) $\vec{a}$, 2.0 km due east (directly toward the east); (b) $\vec{b}$, 2.0 km 30° north of east (at an angle of 30° toward the north from due east); (c) $\vec{c}$, 1.0 km due west. Alternatively, you may substitute either $-\vec{b}$ for $\vec{b}$ or $-\vec{c}$ for $\vec{c}$. What is the greatest distance you can be from base camp at the end of the third displacement?

Reasoning: Using a convenient scale, we draw vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $-\vec{b}$, and $-\vec{c}$ as in Fig. 3-7a. We then mentally slide the vectors over the page, connecting three of them at a time in head-to-tail arrangements to find their vector sum $\vec{d}$. The tail of the first vector represents base camp. The head of the third vector represents the point at which you stop. The vector sum $\vec{d}$ extends from the tail of the first vector to the head of the third vector. Its magnitude d is your distance from base camp.

We find that distance d is greatest for a head-to-tail arrangement of vectors $\vec{a}$, $\vec{b}$, and $-\vec{c}$. They can be in any order, because their vector sum is the same for any order.
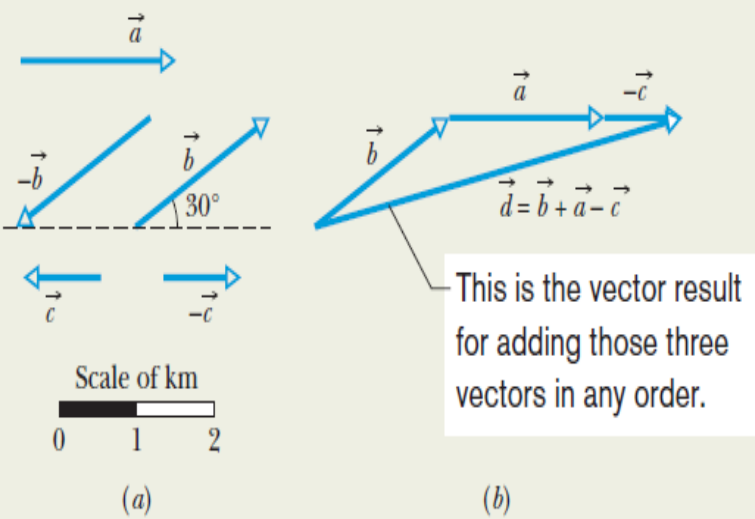

This is the vector result for adding those three vectors in any order.

Fig. 3-7 (a) Displacement vectors; three are to be used. (b) Your distance from base camp is greatest if you undergo displacements $\vec{a}$, $\vec{b}$, and $-\vec{c}$, in any order.

The order shown in Fig. 3-7b is for the vector sum

$$\vec{d} = \vec{b} + \vec{a} + (-\vec{c}).$$

Using the scale given in Fig. 3-7a, we measure the length d of this vector sum, finding

$$d = 4.8 \text{ m.} \qquad \text{(Answer)}$$

3.4 Components of vectors

The component of a vector along an axis is the projection of the vector onto that axis.

The process of finding the components of a vector is called resolution of the vector.

In 3-dimensions, there are three components of a vector along pre-defined x-, y-, and z-axes.
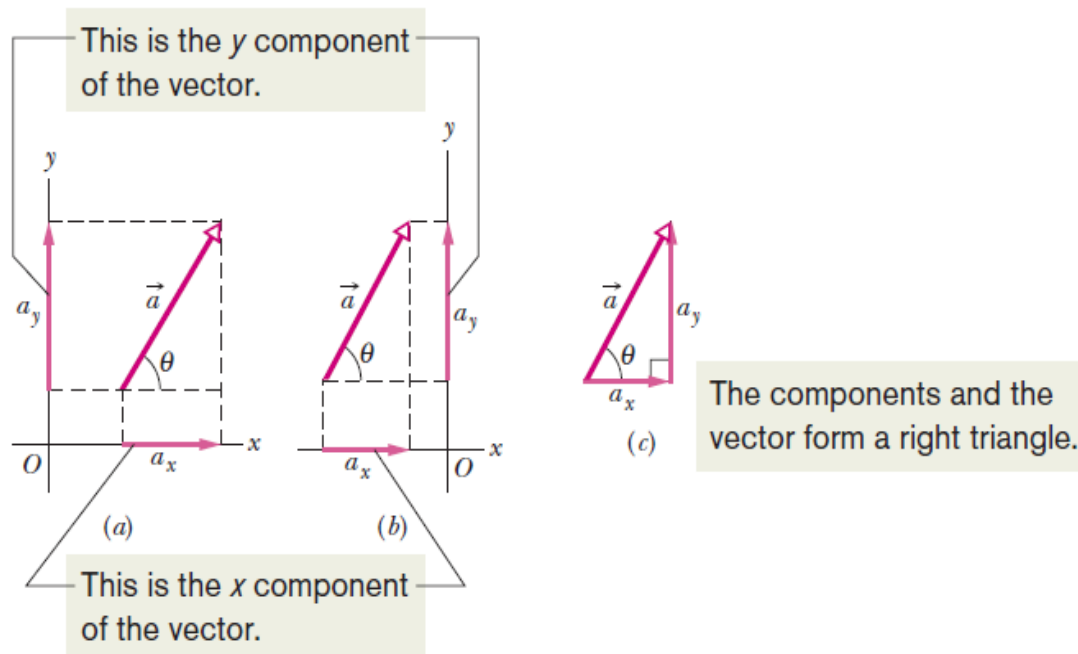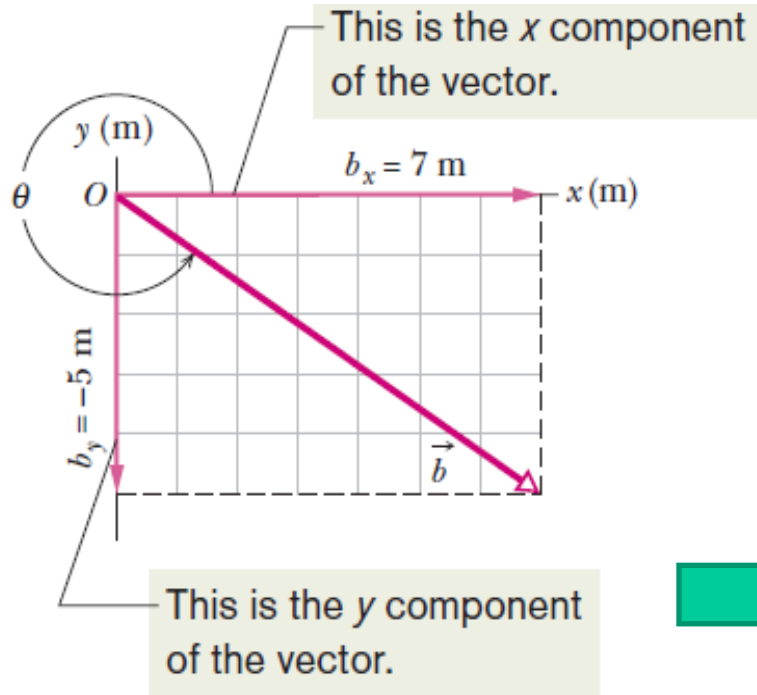

Fig. 3-8 (a) The components a_x and a_y of vector $\vec{a}$. (b) The components are unchanged if the vector is shifted, as long as the magnitude and orientation are maintained. (c) The components form the legs of a right triangle whose hypotenuse is the magnitude of the vector.

3.4 Components of vectors

We find the components of a vector by using the right triangle rules.

This is the *x* component of the vector.

This is the *y* component of the vector.

$b_x = 7 \text{ m}$

$b_y = -5 \text{ m}$

y (m)

x (m)

θ *O*

$\vec{b}$

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x}$$

Fig. 3-9 The component of $\vec{b}$ on the *x* axis is positive, and that on the *y* axis is negative.

Example: vectors

A small airplane leaves an airport on an overcast day and is later sighted 215 km away, in a direction making an angle of 22° east of due north. How far east and north is the airplane from the airport when sighted?

KEY IDEA

We are given the magnitude (215 km) and the angle (22° east of due north) of a vector and need to find the components of the vector.

Calculations: We draw an xy coordinate system with the positive direction of x due east and that of y due north (Fig. 3-10). For convenience, the origin is placed at the airport. The airplane's displacement $\vec{d}$ points from the origin to where the airplane is sighted.

To find the components of $\vec{d}$, we use Eq. 3-5 with $\theta = 68°\ (= 90° - 22°)$:

$$d_x = d \cos \theta = (215 \text{ km})(\cos 68°)$$
$$= 81 \text{ km} \qquad \text{(Answer)}$$

$$d_y = d \sin \theta = (215 \text{ km})(\sin 68°)$$
$$= 199 \text{ km} \approx 2.0 \times 10^2 \text{ km.} \qquad \text{(Answer)}$$

Thus, the airplane is 81 km east and 2.0×10^2 km north of the airport.
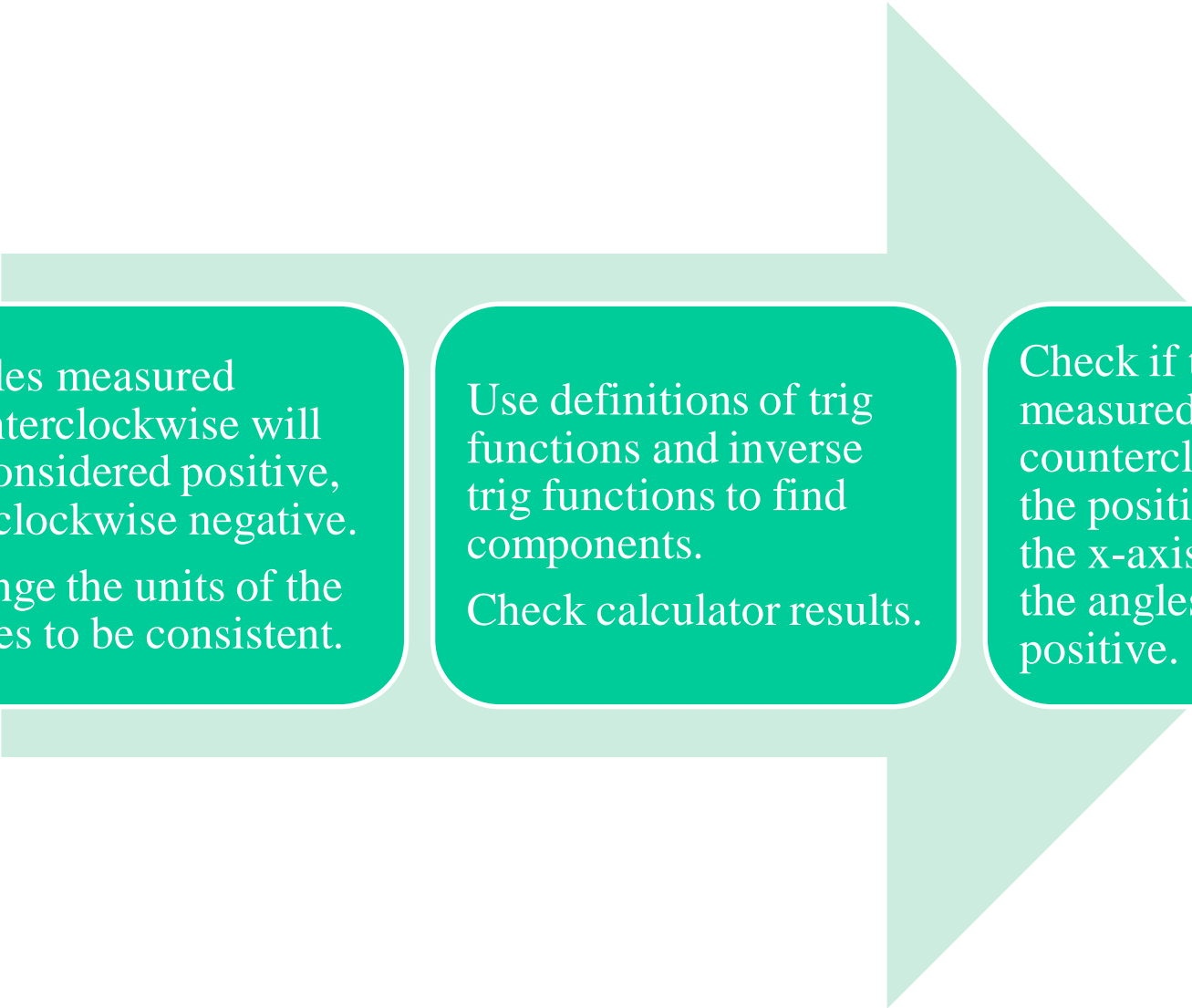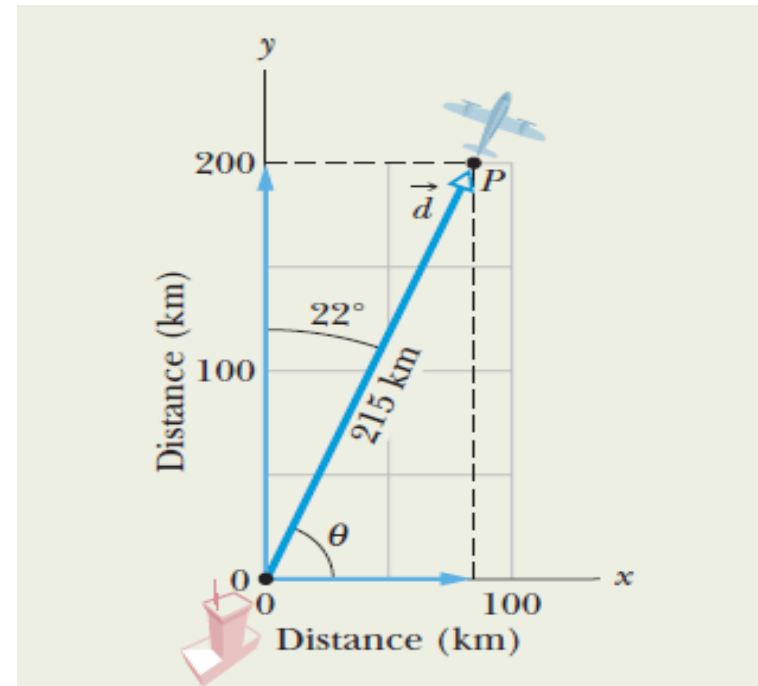

Fig. 3-10 A plane takes off from an airport at the origin and is later sighted at P.

3.4: Problem Solving Check-points

Angles measured counterclockwise will be considered positive, and clockwise negative.

Change the units of the angles to be consistent.

Use definitions of trig functions and inverse trig functions to find components.

Check calculator results.

Check if the angles are measured counterclockwise from the positive direction of the x-axis, in which case the angles will be positive.

3.5: *Unit vectors*

A unit vector is a vector of unit magnitude, pointing in a particular direction.

Unit vectors pointing in the x-, y-, and z-axes are usually designated by

$$\hat{i}, \hat{j}, \hat{k}$$ respectively

Therefore vector, $$\vec{a}$$

with components a_x and a_y in the x- and y-directions,

can be written in terms of the following vector sum:

$$\vec{a} = a_x \hat{i} + a_y \hat{j}$$
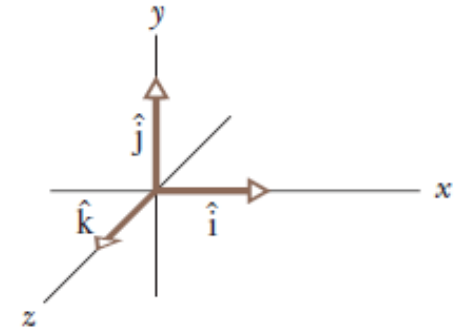
The unit vectors point along axes.

Fig. 3-13 Unit vectors $\hat{i}, \hat{j},$ and $\hat{k}$ define the directions of a right-handed coordinate system.

3.6: Adding vectors by components

If

$$\vec{r} = \vec{a} + \vec{b},$$

then

$$r_x = a_x + b_x$$
$$r_y = a_y + b_y$$
$$r_z = a_z + b_z.$$

Therefore, two vectors must be equal if their corresponding components are equal.

The procedure of adding vectors also applies to vector subtraction.

Therefore, $\vec{d} = \vec{a} - \vec{b}$ ⟹ $d_x = a_x - b_x, \quad d_y = a_y - b_y, \quad \text{and} \quad d_z = a_z - b_z,$

Where, $\vec{d} = d_x \hat{i} + d_y \hat{j} + d_z \hat{k}.$

Basic Vector Operations: http://hyperphysics.phy-astr.gsu.edu/hbase/vect.html#veccon

Example: vector addition

The desert ant *Cataglyphis fortis* lives in the plains of the Sahara desert. When one of the ants forages for food, it travels from its home nest along a haphazard search path, over flat, featureless sand that contains no landmarks. Yet, when the ant decides to return home, it turns and then runs directly home. According to experiments, the ant keeps track of its movements along a mental coordinate system. When it wants to return to its home nest, it effectively sums its displacements along the axes of the system to calculate a vector that points directly home. As an example of the calculation, let's consider an ant making five runs of 6.0 cm each on an *xy* coordinate system. in the directions shown in Fig. 3-16a, starting from home. At the end of the fifth run, what are the magnitude and angle of the ant's net displacement vector $\vec{d}_{net}$, and what are those of the homeward vector $\vec{d}_{home}$ that extends from the ant's final position back to home? In a real situation, such vector calculations might involve thousands of such runs.

KEY IDEAS

(1) To find the net displacement $\vec{d}_{net}$, we need to sum the five individual displacement vectors:

$$\vec{d}_{net} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4 + \vec{d}_5.$$

(2) We evaluate this sum for the *x* components alone,

$$d_{net,x} = d_{1x} + d_{2x} + d_{3x} + d_{4x} + d_{5x},$$

and for the *y* components alone,

$$d_{net,y} = d_{1y} + d_{2y} + d_{3y} + d_{4y} + d_{5y}.$$

(3) We construct $\vec{d}_{net}$ from its *x* and *y* components.

Calculations:

$d_{1x} = (6.0 \text{ cm}) \cos 0° = +6.0 \text{ cm}$

$d_{2x} = (6.0 \text{ cm}) \cos 150° = -5.2 \text{ cm}$

$d_{3x} = (6.0 \text{ cm}) \cos 180° = -6.0 \text{ cm}$

$d_{4x} = (6.0 \text{ cm}) \cos(-120°) = -3.0 \text{ cm}$

$d_{5x} = (6.0 \text{ cm}) \cos 90° = 0.$

$d_{net,x} = +6.0 \text{ cm} + (-5.2 \text{ cm}) + (-6.0 \text{ cm}) + (-3.0 \text{ cm}) + 0$

$\qquad = -8.2 \text{ cm}.$

$d_{net,y} = +3.8 \text{ cm}.$

TABLE 3-1

Run	d_x (cm)	d_y (cm)
1	+6.0	0
2	−5.2	+3.0
3	−6.0	0
4	−3.0	−5.2
5	0	+6.0
net	−8.2	+3.8

Vector $\vec{d}_{net}$ and its *x* and *y* components are shown in Fig. 3-16b. To find the magnitude and angle of $\vec{d}_{net}$ from its components, we use Eq. 3-6. The magnitude is

$$d_{net} = \sqrt{d_{net,x}^2 + d_{net,y}^2}$$
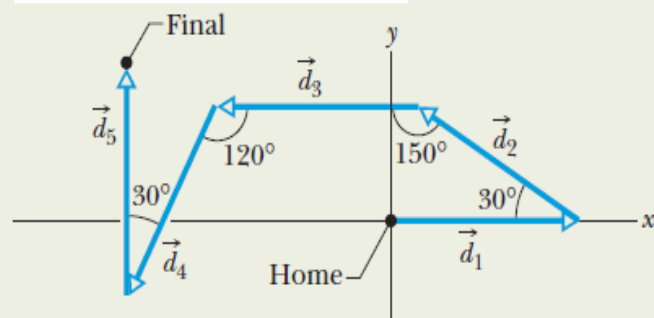
$$= \sqrt{(-8.2 \text{ cm})^2 + (3.8 \text{ cm})^2} = 9.0 \text{ cm}.$$

To find the angle (measured from the positive direction of *x*), we take an inverse tangent:

$$\theta = \tan^{-1}\left(\frac{d_{net,y}}{d_{net,x}}\right)$$

$$= \tan^{-1}\left(\frac{3.8 \text{ cm}}{-8.2 \text{ cm}}\right) = -24.86°.$$
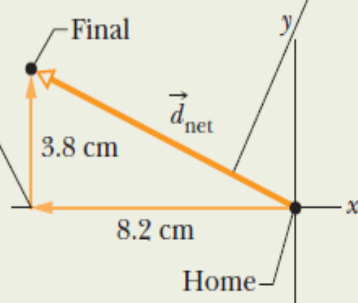
Example: vector addition

NOTE:

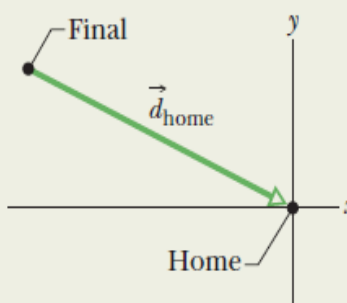To add these vectors, find their net x component and their net y component.

(a)

Then arrange the net components head to tail.

This is the result of the addition.

3.8 cm

8.2 cm

(b)

(c)

Caution: Taking an inverse tangent on a calculator may not give the correct answer. The answer $-24.86°$ indicates that the direction of $\vec{d}_{net}$ is in the fourth quadrant of our xy coordinate system. However, when we construct the vector from its components (Fig. 3-16b), we see that the direction of $\vec{d}_{net}$ is in the second quadrant. Thus, we must "fix" the calculator's answer by adding 180°:

$$\theta = -24.86° + 180° = 155.14° \approx 155°.$$

Thus, the ant's displacement $\vec{d}_{net}$ has magnitude and angle

$$d_{net} = 9.0 \text{ cm at } 155°. \qquad \text{(Answer)}$$

Vector $\vec{d}_{home}$ directed from the ant to its home has the same magnitude as $\vec{d}_{net}$ but the opposite direction (Fig. 3-16c). We already have the angle $(-24.86° \approx -25°)$ for the direction opposite $\vec{d}_{net}$. Thus, $\vec{d}_{home}$ has magnitude and angle

$$d_{home} = 9.0 \text{ cm at } -25°. \qquad \text{(Answer)}$$

A desert ant traveling more than 500 m from its home will actually make thousands of individual runs. Yet, it somehow knows how to calculate $\vec{d}_{home}$ (without studying this chapter).

3.7: Vectors and the Laws of Physics

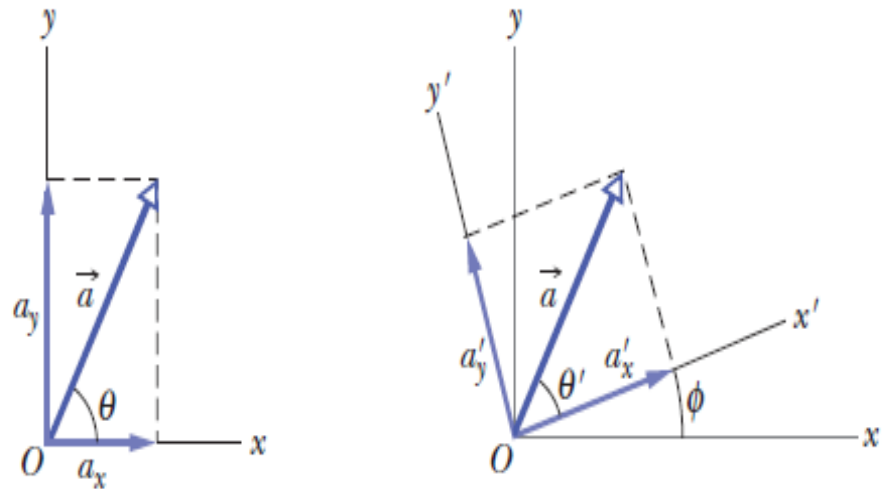**Freedom of choosing a coordinate system:**

Relations among vectors
do not depend on the
origin or the orientation
of the axes.

Relations in physics are
also independent of the
choice of the coordinate
system.

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_x'^2 + a_y'^2}$$

$$\theta = \theta' + \phi.$$

Rotating the axes
changes the components
but not the vector.

A. Multiplying a vector by a scalar

Multiplying a vector by a scalar changes the magnitude but not the direction:

$$\vec{a} \, x \, s = s\vec{a}$$

B. Multiplying a vector by a vector: Scalar (Dot) Product

The scalar product between two vectors is written as:

$$\vec{a} \cdot \vec{b}$$

It is defined as:

$$\vec{a} \cdot \vec{b} = ab \cos \phi,$$

Here, a and b are the magnitudes of vectors a and b respectively, and f is the angle between the two vectors.
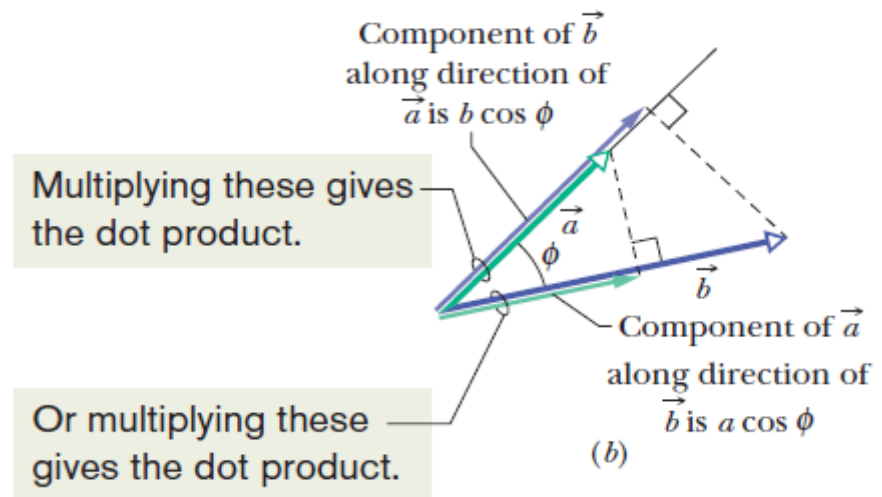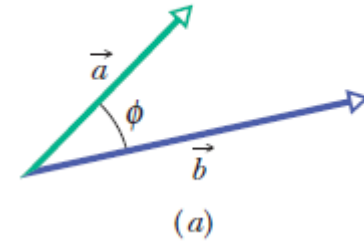The right hand side is a scalar quantity.

Component of $\vec{b}$ along direction of $\vec{a}$ is $b \cos \phi$

Multiplying these gives the dot product.

Or multiplying these gives the dot product.

Component of $\vec{a}$ along direction of $\vec{b}$ is $a \cos \phi$

(a)

(b)

Fig. 3-18 (a) Two vectors $\vec{a}$ and $\vec{b}$, with an angle ϕ between them. (b) Each vector has a component along the direction of the other vector.

C. Multiplying a vector with a vector: Vector (Cross) Product

The vector product between two vectors a and b can be written as: $\vec{a} \times \vec{b}$.

The result is a new vector c, which is:

$$c = ab \sin \phi,$$

Here a and b are the magnitudes of vectors **a** and **b** respectively, and f is the smaller of the two angles between **a** and **b** vectors.

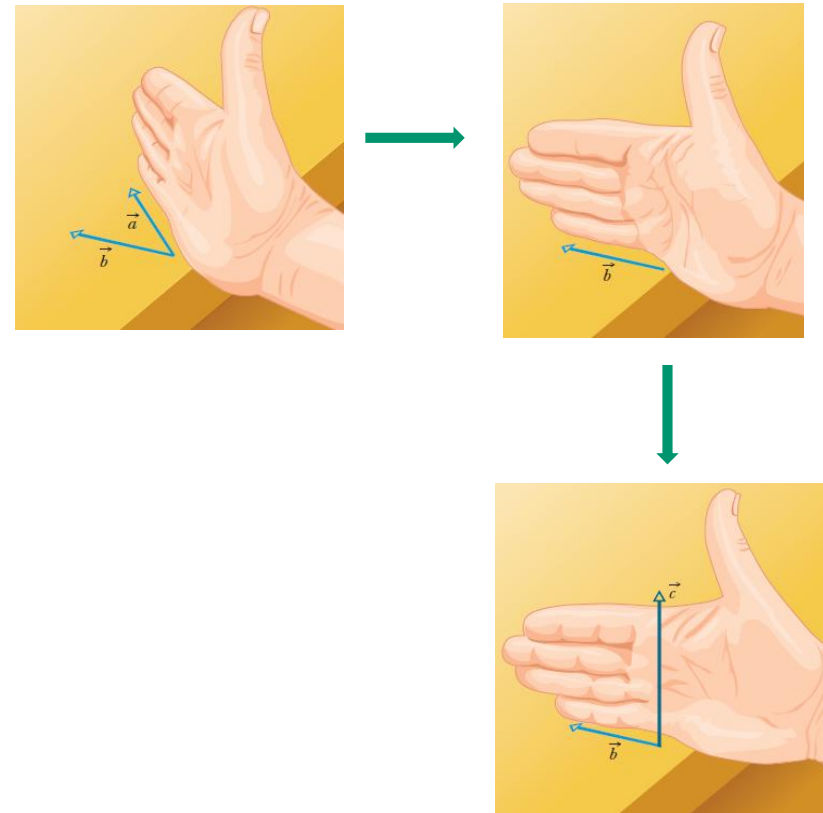The right-hand rule allows us to find the direction of vector c.

Fig. 3-19 Illustration of the right-hand rule for vector products. (*a*) Sweep vector $\vec{a}$ into vector $\vec{b}$ with the fingers of your right hand. Your outstretched thumb shows the direction of vector $\vec{c} = \vec{a} \times \vec{b}$.

3.8: Multiplying vectors; vector product in unit-vector notation:

$$\vec{a} \times \vec{b} = (a_x \hat{i} + a_y \hat{j} + a_z \hat{k}) \times (b_x \hat{i} + b_y \hat{j} + b_z \hat{k})$$

$$= (a_y b_z - b_y a_z)\hat{i} + (a_z b_x - b_z a_x)\hat{j} + (a_x b_y - b_x a_y)\hat{k}.$$

Note that:

$$a_x \hat{i} \times b_x \hat{i} = a_x b_x (\hat{i} \times \hat{i}) = 0,$$

And,

$$a_x \hat{i} \times b_y \hat{j} = a_x b_y (\hat{i} \times \hat{j}) = a_x b_y \hat{k}$$

Sample problem: vector product

In Fig. 3-20, vector $\vec{a}$ lies in the xy plane, has a magnitude of 18 units and points in a direction 250° from the positive direction of the x axis. Also, vector $\vec{b}$ has a magnitude of 12 units and points in the positive direction of the z axis. What is the vector product $\vec{c} = \vec{a} \times \vec{b}$?
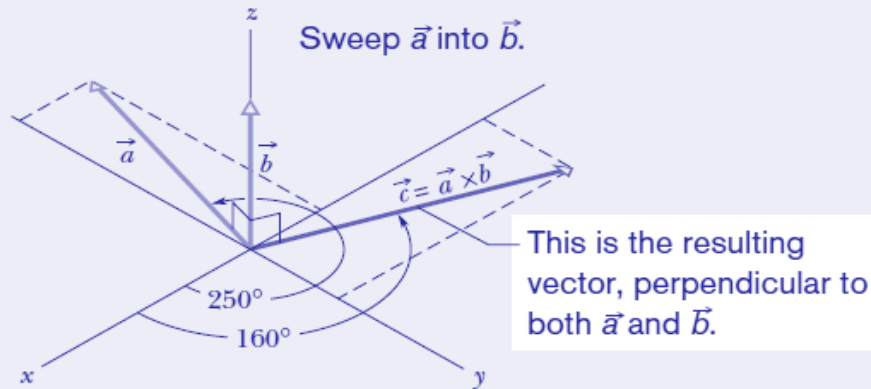
Calculations: For the magnitude we write

$$c = ab \sin \phi = (18)(12)(\sin 90°) = 216. \qquad \text{(Answer)}$$

To determine the direction in Fig. 3-20, imagine placing the fingers of your right hand around a line perpendicular to the plane of $\vec{a}$ and $\vec{b}$ (the line on which $\vec{c}$ is shown) such that your fingers sweep $\vec{a}$ into $\vec{b}$. Your outstretched thumb then

Sweep $\vec{a}$ into $\vec{b}$.

This is the resulting vector, perpendicular to both $\vec{a}$ and $\vec{b}$.

gives the direction of $\vec{c}$. Thus, as shown in the figure, $\vec{c}$ lies in the xy plane. Because its direction is perpendicular to the direction of $\vec{a}$ (a cross product always gives a perpendicular vector), it is at an angle of

$$250° - 90° = 160° \qquad \text{(Answer)}$$

from the positive direction of the x axis.

Fig. 3-20 Vector $\vec{c}$ (in the xy plane) is the vector (or cross) product of vectors $\vec{a}$ and $\vec{b}$.

KEY IDEA

When we have two vectors in magnitude-angle notation, we find the magnitude of their cross product with Eq. 3-27 and the direction of their cross product with the right-hand rule of Fig. 3-19.

Sample problem: vector product, unit vector notation

If $\vec{a} = 3\hat{i} - 4\hat{j}$ and $\vec{b} = -2\hat{i} + 3\hat{k}$, what is $\vec{c} = \vec{a} \times \vec{b}$?

KEY IDEA

When two vectors are in unit-vector notation, we can find their cross product by using the distributive law.

Calculations: Here we write

$$\vec{c} = (3\hat{i} - 4\hat{j}) \times (-2\hat{i} + 3\hat{k})$$
$$= 3\hat{i} \times (-2\hat{i}) + 3\hat{i} \times 3\hat{k} + (-4\hat{j}) \times (-2\hat{i})$$
$$+ (-4\hat{j}) \times 3\hat{k}.$$

We next evaluate each term with Eq. 3-27, finding the direction with the right-hand rule. For the first term here, the angle ϕ between the two vectors being crossed is 0. For the other terms, ϕ is 90°. We find

$$\vec{c} = -6(0) + 9(-\hat{j}) + 8(-\hat{k}) - 12\hat{i}$$
$$= -12\hat{i} - 9\hat{j} - 8\hat{k}. \qquad \text{(Answer)}$$

This vector $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, a fact you can check by showing that $\vec{c} \cdot \vec{a} = 0$ and $\vec{c} \cdot \vec{b} = 0$; that is, there is no component of $\vec{c}$ along the direction of either $\vec{a}$ or $\vec{b}$.